DISCRETE MATHEMATICS

The n! conjecture and a vector bundle on the Hilbert scheme of n points in the plane

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Abstract

The focus of this paper is on algebraic vector bundles over $\mathbb{P}^n$ and their applications to the Garsia–Haiman representation theoretic interpretation of the Macdonald symmetric polynomials. This interpretation involves a certain bigraded S_n-module, $\boldsymbol{H}_\mu$, indexed by partitions μ of n. Bergeron and Garsia (preprint) consider the relationships between $\boldsymbol{H}_\mu$, for μ a partition of $n+1$, and the spaces $\boldsymbol{H}_{\mu_i}$, for μ_i a partition of n contained in μ. They formulated conjectures regarding the sums and intersections of these spaces. This paper provides a geometric interpretation of these conjectures. © 2000 Elsevier Science B.V. All rights reserved.

Résumé

Dans cet article on met l'accent sur les fibrés vectoriels (algébriques) sur $\mathbb{P}^n$ et leur application à l'interprétation en termes de réprésentations des polynômes symétriques de Macdonald, due à Garsia–Haiman. Cette interprétation utilise certains S_n-modules bigradués, $\boldsymbol{H}_\mu$, indexés par les partitions μ of n. La relation entre $\boldsymbol{H}_\mu$ pour une partition μ de $n+1$ et les espaces $\boldsymbol{H}_{\mu_i}$ pour les partitions μ_i de n incluses dans μ a été étudiée par. Bergeron et Garsia dans (preprint), où ils proposent des conjectures concernant la somme et l'intersection de tels espaces. Dans le présent article on fournit une interprétation géométrique de ces conjectures. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

Macdonald introduced a two-parameter family of symmetric functions, $J_\mu(\boldsymbol{x};q,t)$, now known as the *integral form* Macdonald polynomials. Expanding in terms of certain modified Schur symmetric functions, we obtain the *Macdonald q, t-Kostka coefficients*

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$K_{\lambda_\mu}(q,t)$:

$$J_\mu(\boldsymbol{x};\ q,t)=\sum_\lambda K_{\lambda\mu}(q,t)s_\lambda[X(1-t)].$$

The notation $s_\lambda[X(1-t)]$ is defined in Section 2. One of the major open problems regarding the Macdonald polynomials is to show that the $K_{\lambda\mu}(q,t)$, which are polynomials in the two parameters, have non-negative integer coefficients. This is known as Macdonald's positivity conjecture.

In order to find a combinatorial and representation theoretic interpretation of the positivity conjecture, Garsia and Haiman [4] define a doubly graded S_n module, $\boldsymbol{H}_\mu$, indexed by a partition of n. They conjecture that the dimension of this module is $n!$, an assertion known as the $n!$ conjecture. Haiman [10] has shown that the $n!$ conjecture implies Macdonald's positivity conjecture. Specifically, he shows that if the dimension of $\boldsymbol{H}_\mu$ is $n!$, then its characteristic series (the generating function giving the characters and multiplicities in each bigraded component) is a simple transformation of the Macdonald polynomial indexed by μ.

Many computer experiments and some partial results point toward the validity of the $n!$ conjecture. With the intention of proving the $n!$ conjecture by induction, Bergeron and Garsia [1] consider the relationship between $\boldsymbol{H}_\mu$, for μ a partition of $n+1$, and the spaces $\boldsymbol{H}_{\mu_i}$, for μ_i a partition of n contained in μ. They conjecture that the space $\boldsymbol{H}_\mu$ has a particular decomposition and the pieces of this decomposition come from the lattice of spaces generated by the $\boldsymbol{H}_{\mu_i}$'s according to a rule due to Bergeron and Haiman. As a result, Bergeron and Garsia develop conjectures regarding the sums and intersections of the spaces $\boldsymbol{H}_{\mu_i}$.

This paper gives an explanation of these conjectures of Bergeron and Garsia in an algebraic geometric setting that was originally proposed by Haiman. In Section 3, we describe the geometric interpretation in the context of the Hilbert scheme of n points in the plane. We construct a coherent sheaf $\mathscr{P}$ on the Hilbert scheme and show that the $n!$ conjecture is true if and only if $\mathscr{P}$ is a locally free sheaf, i.e., a vector bundle (necessarily of rank $n!$). Since the intention is ultimately to prove the $n!$ conjecture for $n+1$ by induction, we assume the $n!$ conjecture for partitions of n. We then work with the fact that the fibers of $\mathscr{P}$ over special points are isomorphic to the Garsia–Haiman modules, $\boldsymbol{H}_{\mu_i}$.

Section 4 outlines our main geometric results. We study the restriction of $\mathscr{P}$ to subvarieties isomorphic to $\mathbb{P}^{k-1}$ contained in the Hilbert scheme. We consider this to be the geometric analog of the collection of the spaces $\boldsymbol{H}_{\mu_i}$, when the given partition μ has k corners. We reduce the series of conjectures of Bergeron and Garsia to one simple conjecture on the structure of this vector bundle, restricted to a projective space $\mathbb{P}^k$ embedded in the Hilbert scheme. We finally reinterpret geometric statements combinatorially.

2. Background

We begin this section by giving a definition of a slightly non-traditional version of the Macdonald polynomials. Then we proceed to give a description of the Garsia–Haiman S_n module and the inductive approach of Bergeron and Garsia.

2.1. Macdonald Polynomials

Our notation will be compatible with that of Macdonald in [13]. We let $\mu \vdash n$ be the notation for μ a partition of n, where $\mu = (\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_l > 0)$, and μ' denotes its conjugate.

First, we recall the plethystic notation used by Garsia and Haiman [4]. Let A be a formal power series in the variables, $\{a_1, a_2, \ldots\}$. Define $p_k[A]$ to be the expression that replaces each a_i by a_i^k. Then we extend so that $p_k \mapsto p_k[A]$ is a ring homomorphism on the whole ring of symmetric functions. For example, if $X = x_i + x_2 + \cdots$, then $f[X] = f(x_1, x_2, \ldots)$ and $f[X/(1-q)] = f(x_1, x_2, \ldots, qx_1, qx_2, \ldots)$.

Macdonald [13] introduced a basis for the ring of symmetric functions, $\{J_\mu(\boldsymbol{x}, q, t)\}$, in the variables $\boldsymbol{x} = \{x_1, x_2, \ldots\}$ indexed by partitions $\mu \vdash n$, with coefficients in the field of rational functions $\mathbb{Q}(q, t)$. Expanding these in terms of modified Schur functions, we obtain

$$J_\mu[X;\ q,t] = \sum_\lambda K_{\lambda\mu}(q,t) s_\lambda[X(1-t)]. \tag{2.1}$$

The coefficients $K_{\lambda\mu}(q,t)$, which are called Macdonald's q,t-Kostka coefficients, are polynomials in the two parameters with integer coefficients. This was proved independently by various authors [6,7,11,12,14]. Macdonald's positivity conjecture states that these polynomials have non-negative integer coefficients.

In this paper, we refer to the variant basis, first introduced in [4],

$$\tilde{H}_\mu[X;\ q,t] = t^{n(\mu)} J_\mu\left[\frac{X}{1-t^{-1}}; q, t^{-1}\right], \tag{2.2}$$

where $n(\mu) = \sum_i (i-1)\mu_i$. Then we have

$$\tilde{H}_\mu[X;\ q,t] = \sum_\lambda \tilde{K}_{\lambda\mu}(q,t) s_\lambda[X] \tag{2.3}$$

with

$$\tilde{K}_{\lambda\mu}(q,t) = t^{n(\mu)} K_{\lambda\mu}(q, t^{-1}). \tag{2.4}$$

2.2. The $n!$ conjecture

Garsia and Haiman reformulate Macdonald's positivity conjecture as the problem of showing that $\tilde{H}_\mu(X;\ q,t)$ is the Frobenius series of some finite-dimensional doubly graded S_n-module.

We recall the definition of the Frobenius series. Let $H = \bigoplus_{h,k} (H)_{h,k}$ be a doubly graded S_n-module. Recall that the Frobenius map, ϕ, is a linear map from S_n characters to the symmetric functions taking the irreducible S_n character χ_λ to the Schur function s_λ. Then the Frobenius series of H is defined to be the generating function for the image by ϕ of the characters in the various bihomogeneous components:

$$\mathscr{F}(H) = \sum_{h,k} \phi(\mathrm{char}(H)_{h,k}) t^h q^k. \tag{2.5}$$

Thus $\mathscr{F}(H)$ is a symmetric function with coefficients in $\mathbb{Q}[q,t]$.

Garsia and Haiman found a candidate for a doubly graded S_n-module that would explain Macdonald's positivity conjecture. The module is defined as follows. Let Δ_μ be the polynomial analogous to the Vandermonde determinant:

$$\Delta_\mu(x_1,\dots,x_n;\, y_1,\dots,y_n) = \det[x_i^{p_j} y_i^{q_j}]_{i,j=1,\dots,n}, \tag{2.6}$$

where the pairs $(p_j,q_j)=(\mathrm{row}(c_j),\mathrm{col}(c_j))$ run through the coordinates of the cells c_j in the Ferrers diagram of μ.

Example. $\mu=(2,1)$

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$$\Delta_{(2,1)}(x_1,\,x_2,\,x_3,\,y_1,\,y_2,\,y_3) = \det\begin{bmatrix} 1 & y_1 & x_1 \\ 1 & y_2 & x_2 \\ 1 & y_3 & x_3 \end{bmatrix}. \tag{2.7}$$

Definition 2.1. $\boldsymbol{H}_\mu$ is the $\mathbb{Q}$-linear span of all partial derivatives of all orders of Δ_μ.

With this module defined, there is the following formulation of the positivity conjecture by Garsia and Haiman.

Conjecture 2.2.

$$\mathscr{F}(\boldsymbol{H}_\mu) = \tilde{H}_\mu(X;\; q,t).$$

Since it is known that $\tilde{K}_{\lambda\mu}(1,1)$ is the dimension of the irreducible representation V_λ, it follows that if the Frobenius series of $\boldsymbol{H}_\mu$ is $\tilde{H}_\mu(X,q,t)$, then $\boldsymbol{H}_\mu$ affords the regular representation of S_n. Thus in particular Conjecture 2.2 implies that the dimension of the space $\boldsymbol{H}_\mu$ is $n!$, a conjecture of Garsia and Haiman known as the $n!$ conjecture. In addition, Haiman proved the following theorem.

Theorem 2.3 (Haiman). *If* $\dim \boldsymbol{H}_\mu = n!$ *then* $\tilde{H}_\mu(X,q,t)$ *is the Frobenius series of* $\boldsymbol{H}_\mu$.

We give an alternate description of $\boldsymbol{H}_\mu$, which is more natural in the algebraic geometry setting. First, we describe the setting in terms of $\mathbb{Q}$. Let $J_\mu \subseteq \mathbb{Q}[x_1,y_1,\dots,x_n,y_n]$ be the ideal of polynomials $p(x_1,y_1,\dots,x_n,y_n)$ with the property that $p(\partial_{x_1},\partial_{y_1},\dots,\partial_{x_n},\partial_{y_n})$ annihilates Δ_μ. Define $R_\mu^{\mathbb{Q}} = \mathbb{Q}[x_1,y_1\dots,x_n,y_n]/J_\mu$ with its natural structure of an S_n module. The ring $R_\mu^{\mathbb{Q}}$ is isomorphic as a doubly graded S_n module to $\boldsymbol{H}_\mu$ see [4] for details. From a geometric point of view it is more natural to work with $R_\mu = \mathbb{C}\otimes R_\mu^{\mathbb{Q}}$, which in any event has the same decomposition as a doubly graded S_n-module.

2.3. Conjectures of Bergeron and Garsia

Let μ be a partition of $(n+1)$ with k corners and let μ_i be the partition of n obtained by removing the ith corner of μ. Bergeron and Garsia studied the relationships that hold between the space $\boldsymbol{H}_\mu$ and the spaces $\boldsymbol{H}_{\mu_i}$. The idea is to construct a basis for $\boldsymbol{H}_\mu$ using information about the $\boldsymbol{H}_{\mu_i}$'s.

In particular, they conjecture together with Bergeron and Haiman that for each cell $(i,j)\in\mu$ there is a particular subspace $\boldsymbol{H}_{ij}$ of $\sum_i \boldsymbol{H}_{\mu_i}$, such that

$$\boldsymbol{H}_\mu = \bigoplus_{(i,j)\in\mu} \boldsymbol{H}_{ij}(\partial)\partial^i_{x_{n+1}}\partial^j_{y_{n+1}}\Delta_\mu, \tag{2.8}$$

where $\boldsymbol{H}_{ij}(\partial)$ is obtained by substituting the partial derivative operator $\partial_{x_p}, \partial_{y_p}$ for the variables x_p, y_p in the polynomials belonging to $\boldsymbol{H}_{ij}$.

Haiman and Bergeron gave a conjectured algorithm for choosing the modules $\boldsymbol{H}_{ij}$. The modules $\boldsymbol{H}_{ij}$ are elements of the lattice of subspaces of $\sum_i \boldsymbol{H}_{\mu_i}$ generated by the spaces $\boldsymbol{H}_{\mu_1},\dots,\boldsymbol{H}_{\mu_k}$. The algorithm is as follows.

First for every (i,j) in the top row of μ, let $\boldsymbol{H}_{ij}=\boldsymbol{H}_{\mu_1}$. The assignments for subsequent rows are obtained inductively by the following procedure. Let A be a row whose assignments have been made, and B be the next row. There are two cases.

1. If the length of B is equal to the length of A, then the assignment for each cell in row B is that of the cell above it in row A.
2. If the lengths of the rows differ, let a be the length of row A, b be the length of row B, and $c=b-a\geqslant 1$. Let

   $$A_1, A_2, \dots, A_a$$

   be the assignments to the cells in row A. Set

   $$A'_s = \begin{cases} A_s & \text{for } 1\leqslant s\leqslant a, \\ \boldsymbol{H}_{\mu_1}+\cdots+\boldsymbol{H}_{\mu_k} & \text{for } s\leqslant 0, \\ 0 & \text{for } a+1\leqslant s\leqslant b. \end{cases} \tag{2.9}$$

Then the sth cell of row B is assigned

$$B_s = A'_s + (A'_{s-c}\cap \boldsymbol{H}_{\mu_i}), \tag{2.10}$$

where the last cell in row B is the ith corner of μ.

Example. For $\mu=(4,3,1)$, the filling is as illustrated in Fig. 1.

To complete the story, Bergeron and Garsia developed conjectures on the lattice of subspaces generated by the spaces $\boldsymbol{H}_{\mu_i}$. The first of these conjectures says the space $\boldsymbol{H}_{\mu_1}+\cdots+\boldsymbol{H}_{\mu_k}$ has a direct sum decomposition in which each space $\boldsymbol{H}_{\mu_i}$ is a partial sum.

Conjecture 2.4. There exists subspaces $\boldsymbol{H}_S$ for $\emptyset\neq S\subseteq\{1,\dots,k\}$ such that

$$\sum_{i=1}^k \boldsymbol{H}_{\mu_i} = \bigoplus \boldsymbol{H}_S \quad\text{and}\quad \boldsymbol{H}_{\mu_i} = \bigoplus_{i\in S} \boldsymbol{H}_S.$$

$\mathbf{H}_{\mu_1}$			
$\mathbf{H}_{\mu_1}+\mathbf{H}_{\mu_2}$	$\mathbf{H}_{\mu_2}$	$\mathbf{H}_{\mu_1}\cap\mathbf{H}_{\mu_2}$	
$\mathbf{H}_{\mu_1}+\mathbf{H}_{\mu_2}+\mathbf{H}_{\mu_3}$	$\mathbf{H}_{\mu_2}+(\mathbf{H}_{\mu_1}\cap\mathbf{H}_{\mu_3})$	$\mathbf{H}_{\mu_2}\cap(\mathbf{H}_{\mu_1}+\mathbf{H}_{\mu_3})$	$\mathbf{H}_{\mu_1}\cap\mathbf{H}_{\mu_2}\cap\mathbf{H}_{\mu_3}$

Fig. 1.

This is equivalent to saying the subspaces $\boldsymbol{H}_{\mu_i}$'s generate a distributive lattice. A further conjecture relates the Frobenius series of the summands. First, let Ψ_i be the operator on symmetric functions with coefficients in $\mathbb{Q}(q,t)$ defined by

$$\Psi_i(f[X;\ q,t]) = \omega f[X;\ q^{-1},t^{-1}]t^{n(\mu_i)}q^{n(\mu_i')}, \tag{2.11}$$

where ω is the involution that sends the Schur function s_λ to $s_{\lambda'}$.

Conjecture 2.5. If $i \in S$ then

$$\mathscr{F}(\boldsymbol{H}_S) = \Psi_I(\mathscr{F}(\boldsymbol{H}_{S^{\mathrm{C}}\cup\{i\}})),$$

where S^{C} is the complement of S in $\{1,\dots,k\}$.

Using the theory of Macdonald polynomials, specifically that the Frobenius series of $\boldsymbol{H}_{\mu_i}$ is given by $\tilde{H}_{\mu_i}[X;\ q,t]$, this conjecture determines the dimensions of the summands of the direct sum decomposition in Conjecture 2.4. In particular, it implies the following conjecture.

Conjecture 2.6. Let $\mu \vdash (n+1)$ with k corners. Let $\mu_1,\mu_2,\dots,\mu_k \vdash n$, where μ_i is μ with the ith corner removed. Then for $\{i_1,\dots,i_m\} \subseteq \{1,\dots,k\}$,

$$\dim \boldsymbol{H}_{\mu_{i_1}} \cap \boldsymbol{H}_{\mu_{i_2}} \cap \cdots \cap \boldsymbol{H}_{\mu_{i_m}} = \frac{n!}{m}.$$

We illustrate with an example. Let each circle in Fig. 2 represent a corner removed module, $\boldsymbol{H}_{\mu_i}$.

Then, we have the decomposition

$$\boldsymbol{H}_{\mu_1}+\boldsymbol{H}_{\mu_2}+\boldsymbol{H}_{\mu_3} = \phi^{100}\oplus\phi^{010}\oplus\phi^{001}\oplus\phi^{101}\oplus\phi^{110}\oplus\phi^{011}\oplus\phi^{111},$$

where, for example,

$$\boldsymbol{H}_{\mu_1} = \phi^{100}\oplus\phi^{101}\oplus\phi^{110}\oplus\oplus\phi^{111}.$$

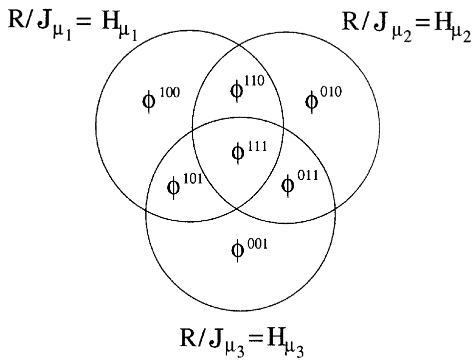

Fig. 2.

We also have an identification of regions in the Venn diagram, for example,

$$\mathscr{F}(\phi^{111}) \cong \Psi_1(\mathscr{F}(\phi^{100})) \cong \Psi_2(\mathscr{F}(\phi^{010})) \cong \Psi_3(\mathscr{F}(\phi^{001}))$$

and

$$\mathscr{F}(\phi^{110})) \cong \Psi_1(\mathscr{F}(\phi^{101})).$$

It can be shown that Conjectures 2.4 and 2.5, together imply that $\dim \boldsymbol{H}_\mu = (n+1)!$. For μ having only 2 corners, Garsia and Bergeron proved Conjectures 2.4 and 2.5, assuming that $\dim \boldsymbol{H}_{\mu_1} \cap \boldsymbol{H}_{\mu_2} = n!/2$ (as predicted by Conjecture 2.6).

Theorem 2.7. *Let* $\mu \vdash (n+1)$ *with two corners. Let* $\mu_1, \mu_2 \vdash n$ *be obtained by removing each corner respectively. Assume that* $\dim \boldsymbol{H}_{\mu_1} = \dim \boldsymbol{H}_{\mu_2} = n!$ *and* $\dim \boldsymbol{H}_{\mu_1} \cap \boldsymbol{H}_{\mu_2} = n!/2$. *Then* $\dim \boldsymbol{H}_\mu = (n+1)!$.

Assuming Conjectures 2.4 and 2.5 along with the $n!$ conjecture, one can write down a 'Pieri rule' for the Macdonald polynomials $\tilde{H}_\mu[X;\ q,t]$ which can be shown remarkably to agree with the known one given by Macdonald.

For our purposes we need to reformulate the statements about the lattice of subspaces generated by the $\boldsymbol{H}_{\mu_i}$'s in terms of the lattice of ideals generated by the J_{μ_i}'s.

Definition 2.8. Let $f, g \in \mathbb{Q}[x_1, y_1, \ldots, x_n, y_n]$. Then $\langle f, g\rangle$ is defined to be the constant term of $f(\partial_{x_1}, \partial_{y_1}, \ldots, \partial_{x_n}, \partial_{y_n}) g(x_1, y_1, \ldots, x_n, y_n)$.

In [5], Garsia and Haiman showed that $J_\mu^\perp \cong \boldsymbol{H}_\mu$ and $(J_\mu^\perp)^\perp \cong J_\mu$. Note that $J_{\mu_1}^\perp \cap J_{\mu_2}^\perp \cong (J_{\mu_1} + J_{\mu_2})^\perp$ and $J_{\mu_1}^\perp + J_{\mu_2}^\perp \cong (J_{\mu_1} \cap J_{\mu_2})^\perp$. Thus, for example, we have with $R^{\mathbb{Q}} = \mathbb{Q}[x_1, y_1, \ldots, x_n, y_n]$, the isomorphisms $R^{\mathbb{Q}}/(J_{\mu_1} \cap \cdots \cap J_{\mu_k}) \cong \boldsymbol{H}_{\mu_1} + \cdots + \boldsymbol{H}_{\mu_k}$ and $R^{\mathbb{Q}}/(J_{\mu_1} + \cdots + J_{\mu_k}) \cong \boldsymbol{H}_{\mu_1} \cap \cdots \cap \boldsymbol{H}_{\mu_k}$. Replacing $R^{\mathbb{Q}}$ by $R = \mathbb{C}[x_1, y_1, \ldots, x_n, y_n]$ and J_{μ_i} by $\mathbb{C} \otimes J_{\mu_i}$ does not alter the characters of these rings as doubly graded S_n modules. Throughout what follows we always work over $\mathbb{C}$.

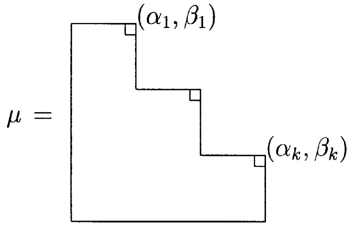

Fig. 3.

3. Geometric interpretation

In an effort to prove the conjectures of Bergeron and Garsia, we translate their statements into an algebraic geometric setting. Assuming the $n!$ conjecture for partitions of n, we construct a vector bundle on the Hilbert scheme of n points in the plane. Using the combinatorics of the global sections of this bundle, we explain the distributive lattice structure. We also give a geometric version of Conjecture 2.5.

We begin by reviewing the description of the Hilbert scheme of n points in the plane (see [9] for details).

3.1. *The Hilbert scheme of n points in the plane*

Let $\mathbb{A}^2 = \operatorname{Spec}\mathbb{C}[x,y]$ be the affine plane. The Hilbert scheme of n points in the plane, $\mathrm{Hilb}^n(\mathbb{A}^2)$, is the set of all ideals $I \subseteq \mathbb{C}[x,y]$ such that $\dim_{\mathbb{C}} \mathbb{C}[x,y]/I = n$. There is a natural action of the torus $T^2 = (\mathbb{C}^*)^2$ on $\mathrm{Hilb}^n(\mathbb{A}^2)$, given by $(t,q)\cdot x = tx$ and $(t,q)\cdot y = qy$. Under this action, an ideal $I \in \mathrm{Hilb}^n(\mathbb{A}^2)$ is a T^2 fixed point if it is of the form

$$I_\mu := (x^i y^j \,|\, (i,j) \notin \mu)$$

for $\mu \vdash n$. Every ideal $I \in \mathrm{Hilb}^n(\mathbb{A}^2)$ has a torus fixed point in the closure of its orbit.

3.2. *Special subvarieties of* $\mathrm{Hilb}^n(\mathbb{A}^2)$

For present purposes, we want to restrict our attention to certain projective subvarieties of the punctual Hilbert scheme. To describe them, let μ be a partition of $(n+1)$ with k corners, i.e. $\mu = (l_1^{m_1}, \ldots, l_1^{m_k})$ where $m_1 l_1 + \cdots + m_k l_k = n+1$. Let (α_i, β_i) be the coordinates of the ith corner cell. See Fig. 3

The geometric analog of the inductive approach would be to consider all the ideals I in $\mathrm{Hilb}^n(\mathbb{A}^2)$ which contain I_μ for this μ. Since $\dim \mathbb{C}[x,y]/I_\mu = n+1$, the only way to obtain such an ideal I of codimension n is if $I = I_\mu + (f)$, where f is a linear combination of corner monomials. Specifically, f must be of the form $f = z_1 x^{\alpha_1} y^{\beta_1} + \cdots + z_k x^{\alpha_k} y^{\beta_k}$, where the z_i's are scalars not all zero. Thus, the ideals $I \in \mathrm{Hilb}^n(\mathbb{A}^2)$ that contain I_μ form a projective space $\mathbb{P}^{k-1} \subseteq \mathrm{Hilb}^n(\mathbb{A}^2)$ with coordinates $(z_1 : \cdots : z_k)$. For I in this family, we use the notation

$$I(z_1 : \cdots : z_k) = (I_\mu + (z_1 x^{\alpha_1} y^{\beta_1} + \cdots + z_k x^{\alpha_k} y^{\beta_k})). \tag{3.1}$$

We note that $I_{\mu_i} = I_\mu + (x^{\alpha_i} y^{\beta_i}) = I(0 : \cdots : 1 : \cdots : 0)$ with 1 in position i, is the ideal that corresponds to the predecessor of μ_i of μ.

3.3. Constructing our bundle

Let $\mathscr{B}$ be the tautological vector bundle of rank n over $\mathrm{Hilb}^n(\mathbb{A}^2)$, whose fiber at I is the vector space $\mathbb{C}[x,y]/I$. In [9], Haiman proved that the highest exterior power of this tautological bundle is $\mathscr{O}(1)$, for a natural embedding of $\mathrm{Hilb}^n(\mathbb{A}^2)$ as a quasi-projective variety. Note that $\mathscr{B}^{\otimes n}$ can be identified as the bundle whose fiber at I is $\mathbb{C}[x_1, y_1, \ldots, x_n, y_n]/(I(x_1, y_1) + \cdots + I(x_n, y_n))$. Note that $\mathscr{B}$ and $\mathscr{B}^{\otimes n}$ are bundles of $\mathbb{C}$-algebras.

There is a natural action of S_n on $\mathscr{B}^{\otimes n}$ which permutes the factors. We then have the map:

$$\begin{aligned} &\mathscr{B}^{\otimes n} \otimes \mathscr{B}^{\otimes n} \to \mathscr{B}^{\otimes n} \to \wedge^n \mathscr{B} \cong \mathscr{O}(1), \\ &f \otimes g \mapsto fg \mapsto \mathrm{Alt}(fg), \end{aligned} \tag{3.2}$$

where the first map is multiplication and the second is the S_n alternation operator defined by

$$\mathrm{Alt}(f) = \frac{1}{n!} \sum_{w \in S_n} \varepsilon(w) w(f). \tag{3.3}$$

This induces a map:

$$\phi : \mathscr{B}^{\otimes n} \to \mathrm{Hom}(\mathscr{B}^{\otimes n}, \wedge^n \mathscr{B}) \cong (\mathscr{B}^{\otimes n})^* \otimes \mathscr{O}(1). \tag{3.4}$$

Let $\mathscr{J} := \ker \phi$. Define $\mathscr{P}$ to be $\mathscr{B}^{\otimes n}/\mathscr{J}$, or equivalently, $\mathrm{im}\,\phi$. A priori, the subsheaf $\mathscr{J}$ might not be subbundle and thus $\mathscr{P}$ is only a coherent sheaf. We show that the $n!$ conjecture holds for partitions of n if and only $\mathscr{P}$ is locally free, i.e., $\mathscr{J}$ is a subbundle and $\mathscr{P}$ is the corresponding quotient bundle.

Lemma 3.1. *Let $\phi_I : \mathscr{B}_I^{\otimes n} \to ((\mathscr{B}^{\otimes n})^* \otimes \mathscr{O}(1))_I$ be the map indued on the fiber at I by* (3.4). *Let $I \in \mathrm{Hilb}^n(\mathbb{A}^2)$ be a generic point, i.e., $I = I(S)$ where $S = \{p_1, \ldots, p_n\} \subseteq \mathbb{A}^2$. Then* $\mathrm{rank}\,\phi_I = n!$.

Proof. The fibers of $\mathscr{B}^{\otimes n}$ at I can be thought of as the set of $\mathbb{C}$-valued functions on the set of sequences $(a_1, \ldots, a_n)$ for $a_i \in S$. If $f \in \mathscr{B}_I^{\otimes n}$ is alternating then f is zero on any sequence with a repeat. Let N be the set of sequences which are permutations of S, i.e., sequences with distinct entries. Thus the complement of N consists of the sequences with repeats.

We claim that $\ker \phi_I = \{f \in \mathscr{B}_I^{\otimes n} : f|_N = 0\}$. Suppose $f|_N = 0$. Then for all $g \in \mathscr{B}_I^{\otimes n}$, $fg|_N = 0$. Thus, $\mathrm{Alt}(fg)|_N = 0$, which implies $f \in \ker \phi_I$. Conversely, suppose

$f|_N \neq 0$. Then choose $\boldsymbol{a} = (a_1, \dots, a_n) \in N$ such that $f(\boldsymbol{a}) \neq 0$. Multiplying f by a suitable g, we obtain a function such that $fg(\boldsymbol{a}) \neq 0$ and $fg(\boldsymbol{b}) = 0$ for all $\boldsymbol{b} \neq \boldsymbol{a}$. Then $\mathrm{Alt}(fg) \neq 0$.

Thus, $\mathscr{B}_I^{\otimes n}/\ker \phi_I$ are exactly the functions on N, which must be $n!$ dimensional; and, in particular this verifies that the rank of ϕ_I is generically $n!$. □

Lemma 3.2. *Let* $\phi_\mu : \mathscr{B}_\mu^{\otimes n} \to ((\mathscr{B}^{\otimes n})^* \otimes \mathscr{O}(1))_\mu$ *be the map induced on the fiber at* I_μ *by* $\mathscr{B}^{\otimes n} \to (\mathscr{B}^{\otimes n})^* \otimes \mathscr{O}(1)$. *Then* $\ker \phi_\mu = J_\mu$. *Thus is particular*, $\operatorname{rank} \phi_\mu = \dim \boldsymbol{H}_\mu$.

Proof. Let $R = \mathbb{C}[x_1, y_1, \dots, x_n, y_n]$. Consider the linear functional

$$l : R \to \Lambda^n \mathscr{B}_\mu \cong \mathbb{C} \tag{3.5}$$

given by $l(f) \equiv \mathrm{Alt}(f)$ modulo $I_\mu(x_1, y_1) + \cdots + I_\mu(x_n, y_n)$. The functional kills all f in the nonalternating S_n isotypic components. In addition, it kills every alternation of a monomial except Δ_μ, since a monomial alternant $\mathrm{Alt}(x_1^{h_1} y_1^{k_1} \cdots x_n^{h_n} y_n^{k_n})$ is killed whenever any (h_i, k_i) lies outside the diagram of μ.

Now, this functional l is, up to a scalar multiple, the same as the functional given by $\psi_\mu(f) = \langle f, \Delta_\mu \rangle$, where $\langle\, , \rangle$ is given in Definition 2.9. Then $f \in J_\mu$ if and only if $\langle g, f(\partial_X, \partial_Y)\Delta_\mu \rangle = 0$ for all $g \in R$. But the latter is equivalent to $\langle fg, \Delta_\mu \rangle = 0$ for all $g \in R$. Thus $\psi_\mu(fg) = 0$ for all $g \in R$ if and only if $l(fg) = 0$ for all $g \in R$ which is equivalent to $f \in \ker \phi_\mu$.

In particular, assuming the $n!$ conjecture holds for partition of n, the rank of ϕ_μ is $n!$. □

Corollary 3.3. *Assuming the* $n!$ *conjecture, the fiber of* $\mathscr{P}$ *at the torus fixed point* $I_\mu \in \mathrm{Hilb}^n(\mathbb{A}^2)$ *is isomorphic to*

$$\mathbb{C}[x_1, y_1, \dots, x_n, y_n]/J_\mu,$$

where J_μ *is the ideal of polynomials which annihilates* Δ_μ.

Theorem 3.4. *The* $n!$ *conjecture is true if and only if* $\mathscr{P}$ *is a locally free sheaf or equivalently if the rank of* ϕ *in* (3.4) *is constant.*

Proof. The generic points are dense in $\mathrm{Hilb}^n(\mathbb{A}^2)$, and the generic rank of ϕ_I is $n!$ from Lemma 3.1. Also, every ideal $I \in \mathrm{Hilb}^n(\mathbb{A}^2)$ has a torus fixed point, I_μ in the closure of its orbit [9]. Assuming the $n!$ conjecture, the rank of ϕ at the torus fixed points I_μ is also $n!$. Thus, the rank of ϕ must be constant, and we have that $\mathscr{P}$ is locally free. □

Recall we are assuming by induction that the $n!$ conjecture is true for partitions of n; thus from here on we assume that $\mathscr{P}$ is a bundle. We also mention some properties of $\mathscr{P}$ which will be useful in proving its decomposition when restricted to certain subvarieties of $\mathrm{Hilb}^n(\mathbb{A}^2)$.

Proposition 3.5. *$\mathscr{P}$ has the following properties*:

1. *$\mathscr{P}$ is generated by global sections.*
2. *Assuming the $n!$ conjecture, $\mathscr{P}$ is a vector bundle such that $\mathscr{P} \cong \mathscr{P}^* \otimes \mathscr{O}(1)$ where $\mathscr{P}^*$ is the dual of $\mathscr{P}$.*

Proof. The first part follows from the fact that $\mathscr{P}$ is a quotient of $\mathscr{B}^{\otimes n}$, and $\mathscr{B}^{\otimes n}$ is a quotient of a trivial bundle.

For the second part, observe that the pairing $\mathscr{B}^{\otimes n} \otimes \mathscr{B}^{\otimes n} \to \wedge^n \mathscr{B} \cong \mathscr{O}(1)$ induces a nondegenerate pairing $\mathscr{P} \otimes \mathscr{P} \to \mathscr{O}(1)$. Thus, in particular, $\mathscr{P} \cong \mathrm{Hom}(\mathscr{P}, \mathscr{O}(1)) \cong \mathscr{P}^* \otimes \mathscr{O}(1)$. $\square$

4. Main results

Throughtout this section we assume the $n!$ conjecture for partitions of a given n, and thus also assume that $\mathscr{P}$ is a vector bundle.

We consider the restriction of $\mathscr{P}$ to the subvarieties $\mathbb{P}^{k-1}$. Let $R = \mathbb{C}[x_1, y_1, \ldots, x_n, y_n]$. Since $\mathscr{B}^{\otimes n}$ is the tautological quotient bundle of the trivial sheaf $\mathscr{O} \otimes R$, and $\mathscr{P}$ is a quotient of $\mathscr{B}^{\otimes n}$, we have the following surjective map of vector bundles:

$$\mathscr{O} \otimes R \twoheadrightarrow \mathscr{P} \tag{4.1}$$

and corresponding map on global sections,

$$\theta\colon R \to \Gamma(\mathbb{P}^{k-1}, \mathscr{P}). \tag{4.2}$$

Conjecture 4.1. The map θ given by (4.2) is surjective.

4.1. Two corner case

When μ is a partition with two corners, we are dealing with a projective line $\mathbb{P}^1 \subseteq \mathrm{Hilb}^n(\mathbb{A}^2)$.

Theorem 4.2 (Birkhoff [2], Grothendieck [8]). *Every algebraic vector bundle V over $\mathbb{P}^1$ has a unique decomposition of the form $V \cong \oplus_k \mathscr{O}(k)^{r_k}$.*

Corollary 4.3. $\mathscr{P}|_{\mathbb{P}^1} \cong \mathscr{O}^{n!/2} \oplus \mathscr{O}(1)^{n!/2}$.

Proof. In the decomposition of Theorem 4.2 we cannot have any terms $\mathscr{O}(k)$ and $\mathscr{P}|_{\mathbb{P}^1}$ with $k < 0$, since $\mathscr{P}$ is generated by global sections. Thus, $\mathscr{P} \cong \oplus_{k \geqslant 0} \mathscr{O}(k)^{r_k}$. We also have from the duality property that $\mathscr{P} \cong \mathscr{P}^* \otimes \mathscr{O}(1) \cong \oplus_{k \geqslant 0} \mathscr{O}(-k+1)^{r_k}$. This implies that $r_k = 0$ for all $k \geqslant 2$, and $r_0 = r_1$. Since $\mathscr{P}$ is of rank $n!, r_0 + r_1 = n!$. Thus we obtain the decomposition $\mathscr{P}|_{\mathbb{P}^1} \cong \mathscr{O}^{n!/2} \oplus \mathscr{O}(1)^{n!/2}$. $\square$

We now have the correct setting to prove the following result.

Proposition 4.4. *If the map* θ *of* (4.2) *is surjective, then* $\Gamma(\mathbb{P}^1,\mathscr{P})\cong R/(J_{\mu_1}\cap J_{\mu_2})$.

Proof. Recall we are assuming that $\theta : R \twoheadrightarrow \Gamma(\mathbb{P}^1,\mathscr{P}) = \Gamma(\mathbb{P}^1,\mathcal{O}^{n!/2}\oplus\mathcal{O}(1)^{n!/2})$ is surjective.

First, we note that sections of $\mathcal{O}$ and $\mathcal{O}(1)$ are completely determined by their value at any one point or, respectively, any two points. The sections of $\mathscr{P}|_{\mathbb{P}^1}$ are thus determined by their value on any two points. Given a polynomial $f\in J_{\mu_1}\cap J_{\mu_2}$ (representing a section) then f evaluates to zero in the fibers $\mathscr{P}_{I_{\mu_1}}\cong R/J_{\mu_1}$ and $\mathscr{P}_{I_{\mu_2}}\cong R/J_{\mu_2}$. Since f evaluates to zero at two points, it must be the zero section. Thus $J_{\mu_1}\cap J_{\mu_2}\subseteq\ker\theta$.

Conversely, if $f\in\ker\theta$ then f is the zero section, evaluating to zero in every fiber. In particular, $f\in J_{\mu_1}\cap J_{\mu_2}$. $\square$

In particular from Lemma 4.4, it follows that $\dim(\boldsymbol{H}_{\mu_1}+\boldsymbol{H}_{\mu_2})=\dim R/(J_{\mu_1}\cap J_{\mu_2})=\dim\Gamma(\mathbb{P}^1,\mathscr{P})$, i.e.,

$$\dim(\boldsymbol{H}_{\mu_1}+\boldsymbol{H}_{\mu_2})=\frac{3n!}{2}. \tag{4.3}$$

Furthermore, assuming the inductive hypothesis that $\dim\boldsymbol{H}_{\mu_1}=\dim\boldsymbol{H}_{\mu_2}=n!$, it necessarily follows that

$$\dim(\boldsymbol{H}_{\mu_1}\cap\boldsymbol{H}_{\mu_2})=\frac{n!}{2}. \tag{4.4}$$

4.2. Multiple corner case

In the general case when μ has k corners, we look at the restriction of $\mathscr{P}$ to $\mathbb{P}^{k-1}$. Recall that $\mathbb{P}^{k-1}$ parametrizes all ideals $I\in\mathrm{Hilb}^n(\mathbb{A}^2)$ that contain I_μ in the following way:

$$I(z_1:\cdots:z_k):=(I_\mu+(z_1x^{\alpha_1}y^{\beta_1}+\cdots+z_kx^{\alpha_k}y^{\beta_k})), \tag{4.5}$$

where (α_i,β_i) is the coordinate of a corner cell of μ.

Another description of this particular subvariety of $\mathrm{Hilb}^n(\mathbb{A}^2)$ is the following. Let

$$V=\operatorname*{span}_{\mathbb{C}}\{x^{\alpha_1}y^{\beta_1},\ldots,x^{\alpha_k}y^{\beta_k}\} \tag{4.6}$$

be the linear span of the corner monomials of μ. Note that $\mathbb{P}(V)=\mathbb{P}^{k-1}$. We then have a one to one correspondence between the set of $I\in\mathrm{Hilb}^n(\mathbb{A}^2)$ such that $I_\mu\subseteq I$ and points $L\in\mathbb{P}(V)$, i.e.,

$$I(z_1:\cdots:z_k)\quad\leftrightarrow\quad\text{the line } L=\operatorname*{spin}_{\mathbb{C}}\{z_1x^{\alpha_1}y^{\beta_1}+\cdots+z_kx^{\alpha_k}y^{\beta_k}\}.$$

Let $\mathcal{O}(-1)$ be the tautological bundle on $\mathbb{P}(V)$ whose fiber at $L\in\mathbb{P}(V)$ is L. Let $\mathcal{Q}$ be the quotient bundle whose fiber at L is V/L. Then $\mathcal{Q}$ fits into the exact sequence:

$$0\to\mathcal{O}(-1)\to\mathcal{O}\otimes V\to\mathcal{Q}\to 0, \tag{4.7}$$

where $\mathcal{O}\otimes V$ is a trivial bundle of rank k. We also have for the jth exterior power of $\mathcal{Q}$ the sequence:

$$0\to\mathcal{O}(-1)\otimes\Lambda^{j-1}\mathcal{Q}\to\Lambda^j(\mathcal{O}\otimes V)\to\Lambda^j\mathcal{Q}\to 0. \tag{4.8}$$

We can now give the conjecture regarding the decomposition of $\mathscr{P}|_{\mathbb{P}^{k-1}}$.

Conjecture 4.5 (*The decomposition conjecture*).

$$\mathscr{P}|_{\mathbb{P}^{k-1}} \cong (\Lambda^0 \mathscr{Q})^{n!/k} \oplus \cdots \oplus (\Lambda^i \mathscr{Q})^{n!/k\binom{k-1}{i}} \oplus \cdots \oplus (\Lambda^{k-1} \mathscr{Q})^{n!/k}.$$

Corollary 4.3 implies that the Decomposition Conjecture is true for $\mathbb{P}^1$. In a separate paper [3] we show that a slight strengthening of Conjecture 4.1 would imply the Decomposition Conjecture. Here we will only discuss its implications.

In order to identify the global sections of $\mathscr{P}|_{\mathbb{P}^1}$ with $\boldsymbol{H}_{\mu_1} + \boldsymbol{H}_{\mu_2}$ we made use of its direct sum decomposition and the global sections of the pieces of the direct sum. We must do the same for $\mathscr{P}|_{\mathbb{P}^{k-1}}$. First, we describe the global sections of the conjectured decomposition.

The following well-known fact is a consequence of the Borel–Weil–Bott theorem for line bundles on the flag variety.

Proposition 4.6. $\Gamma(\mathbb{P}^{k-1}, \Lambda^j \mathscr{Q}) \cong \Lambda^j V$.

The next corollary follows immediately from this proposition.

Corollary 4.7. *Assuming the Decomposition Conjecture, we have that*

$$\Gamma(\mathbb{P}^{k-1}, \mathscr{P}) = (\Lambda^0 V)^{n!/k} \oplus \cdots \oplus (\Lambda^i V)^{n!/k\binom{k-1}{i}} \oplus \cdots \oplus (\Lambda^{k-1} V)^{n!/k}. \tag{4.9}$$

We can choose a basis of $\Gamma(\mathbb{P}^{k-1}, \mathscr{P})$ according to the basis on each summand in its direct sum decomposition. Recall V is the linear span of the corner monomials of μ as defined in (4.6). Thus an appropriate basis of V is

$$B_1 = \{b_1, \ldots, b_k \mid b_i = x^{\alpha_i} y^{\beta_i}\}. \tag{4.10}$$

This induces the basis

$$B_j = \{b_{i_1} \wedge \cdots \wedge b_{i_j} \mid i_1 < \cdots < i_j\} \tag{4.11}$$

of $\Lambda^j V$. Note that $\Lambda^0 V \cong \mathbb{C}$ with basis $B_0 = \{1\}$. Fixing the direct sum decomposition in (4.9) and using the basis B_j in each summand we obtain a basis for $\Gamma(\mathbb{P}^{k-1}, \mathscr{P})$.

Next, we describe the fibers of $\mathscr{P}$ with respect to the conjectured decomposition. Set $L_i = \mathrm{span}_{\mathbb{C}}\{x^{\alpha_i} y^{\beta_i}\} = \mathrm{span}_{\mathbb{C}}\{b_i\}$. L_i can be considered as the point in $\mathbb{P}(V)$ which corresponds to $I_{\mu_i} = I_\mu + (x^{\alpha_i} y^{\beta_i}) \in \mathbb{P}^{k-1} \subseteq \mathrm{Hilb}^n(\mathbb{A}^2)$. Then the space V of (4.6) can be identified with $V \cong L_1 \oplus \cdots \oplus L_k$. Furthermore, let $M_i = \oplus_{j \neq 1} L_i$. We can now identify the fibers of $\mathscr{P}$.

1. The fiber of $\mathscr{O} \otimes V$ at I_{μ_i} is identified with $V \cong L_i \oplus M_i$.
2. The fiber of $\mathscr{O}(-1)$ at I_{μ_i} is identified with L_i.
3. The fiber of $\mathscr{Q}$ at I_{μ_i} is identified with $V/L_i \cong M_i$.

4. The fiber of $\Lambda^j\mathscr{Q}$ at I_{μ_i} is identified with $\Lambda^j(M_i)\cong\Lambda^jV/(L_i\otimes\Lambda^{j-1}(M_i))$.
5. Thus, the fiber of $\mathscr{P}$ at I_{μ_i} is identified with

$$P_{I_{\mu_i}}\cong\Lambda^0(M_i)^{n!/k}\oplus\cdots\Lambda^j(M_i)^{n!/k\binom{k-1}{j}}\oplus\cdots\oplus\Lambda^{k-1}(M_i)^{n!/k}. \tag{4.12}$$

We also remark that for any $j<k$,

$$(L_1\otimes\Lambda^{j-1}(M_1))\cap\cdots\cap(L_k\otimes\Lambda^{j-1}(M_k))=0,$$

since Λ^jV is the direct sum of the 1-dimensional spaces $L_{i_1}\otimes\cdots\otimes L_{i_j}$ for $i_1<\cdots<i_j$, and the rth factor in the above intersection consists of those for which $r\in\{i_1,\dots,i_j\}$.

Lemma 4.8. *Assuming the Decomposition Conjecture, the map*

$$i:\Gamma(\mathbb{P}^{k-1},\mathscr{P})\to\mathscr{P}_{I_{\mu_1}}\oplus\cdots\oplus\mathscr{P}_{I_{\mu_k}},$$
$$s\mapsto s(I_{\mu_1})+\cdots+s(I_{\mu_k})$$

is injective.

Proof. If $v\in\Lambda^jV$ (i.e. in $\Gamma(\mathbb{P}^{k-1},\mathscr{P})$) is mapped to zero under i, then

$$v\in(L_1\otimes\Lambda^{j-1}(M_1))\cap\cdots\cap(L_k\otimes\Lambda^{j-1}(M_k)),$$

since it is zero in each of the fibers at $I_{\mu_1},\dots,I_{\mu_k}$. Thus $v=0$. □

We make the following identification of the global sections of $\mathscr{P}$ restricted to $\mathbb{P}^{k-1}$.

Proposition 4.9. *Assuming the Decomposition Conjecture and that θ defined in* (4.2) *is surjective, then*

$$\Gamma(\mathbb{P}^{k-1},\mathscr{P})\cong R/(J_{\mu_1}\cap\cdots\cap J_{\mu_k}).$$

Proof. Recall we are assuming the $\theta:R\twoheadrightarrow\Gamma(\mathbb{P}^{k-1},\mathscr{P})$ is surjective.

Given $f\in R$, if $f\in\ker\theta$ then f is the zero section, evaluating to zero in every fiber. In particular, $f\in J_{\mu_1}\cap\cdots\cap J_{\mu_k}$.

To prove the converse, first we note that from Lemma 4.8 we have that the map

$$i:\Gamma(\mathbb{P}^{k-1},\mathscr{P})\to\mathscr{P}_{I_{\mu_1}}\oplus\cdots\oplus\mathscr{P}_{I_{\mu_k}}$$

is injective. If $f\in J_{\mu_1}\cap\cdots\cap J_{\mu_k}$ then f is zero in each of the fibers $\mathscr{P}_{I_{\mu_1}},\dots,\mathscr{P}_{I_{\mu_k}}$. Thus $f=0$ in $\Gamma(\mathscr{P}^{k-1},\mathscr{P})$, which implies $f\in\ker\theta$. □

We now discuss the distributive lattice structure of the spaces $\boldsymbol{H}_{\mu_i}$. Recall we are still using the assumption that all global sections of $\mathscr{P}|_{\mathbb{P}^{k-1}}$ come from polynomials in R. Thus, we make use of the isomorphism in Proposition 4.9 between the space of global sections, $\Gamma(\mathbb{P}^{k-1},\mathscr{P})$, and $R/(J_{\mu_1}\cap\cdots\cap J_{\mu_k})\cong\boldsymbol{H}_{\mu_1}+\cdots+\boldsymbol{H}_{\mu_k}$. Furthermore, to describe the distributive lattice structure of $\boldsymbol{H}_{\mu_1}+\cdots+\boldsymbol{H}_{\mu_k}$, we prove equivalently, in the algebraic geometric setting, that there is a basis of

$$\Gamma(\mathbb{P}^{k-1},\mathscr{P})\cong R/(J_{\mu_1}\cap\cdots\cap J_{\mu_k})$$

such that each space $J_{\mu_i}/(J_{\mu_1} \cap \cdots \cap J_{\mu_k})$ is spanned by a subset of this basis. This is equivalent since under the surjective map of global sections onto a fiber $\mathscr{P}_{I_{\mu_i}} \cong R/J_{\mu_i}$, i.e.,

$$\begin{array}{ccc} \Gamma(\mathbb{P}^{k-1}, \mathscr{P}) & \xrightarrow{\alpha_i} & \mathscr{P}_{I_{\mu_i}} \\ \downarrow\cong & & \downarrow\cong \\ R/(J_{\mu_1} \cap \cdots \cap J_{\mu_k}) & \xrightarrow{\alpha_i} & R/J_{\mu_i}, \end{array}$$

the kernel of α_i is exactly $J_{\mu_i}/(J_{\mu_1} \cap \cdots \cap J_{\mu_k})$.

Recall that the fiber $\mathscr{P}$ at I_{μ_i} can be identified as

$$\mathscr{P}_{I_{\mu_i}} \cong \mathbb{C}^{n!/k} \oplus \cdots \oplus \Lambda^j(M_i)^{n!/k\binom{k-1}{j}} \oplus \cdots \oplus \Lambda^{k-1}(M_i)^{n!/k},$$

where $M_i = \mathrm{span}_{\mathbb{C}}\{b_j : j \neq i\}$, and b_j is the jth corner monomial as defined earlier in (4.10). We can clearly see that a basis for M_i is $B \setminus \{b_i\}$, where elements of B are all the corner monomials. For $\Gamma(\mathbb{P}^{k-1}, \mathscr{P})$, we use the basis chosen earlier. Thus, looking at the image of our chosen basis elements of $\Gamma(\mathbb{P}^{k-1}, \mathscr{P})$ under the map α_i above, we see that any exterior products involving b_i must be in the kernel of α_i i.e., must be in $J_{\mu_i}/(J_{\mu_1} \cap \cdots \cap J_{\mu_k})$. Since dim R/J_{μ_i} is assumed to be $n!$ and the number of basis elements in $\Gamma(\mathbb{P}^{k-1}, \mathscr{P})$ involving a b_i is equal to the dim $\Gamma(\mathbb{P}^{k-1}, \mathscr{P}) - n!$, these basis elements involving a b_i must span $J_{\mu_i}/(J_{\mu_1} \cap \cdots \cap J_{\mu_k})$. Thus we arrive at the distributive lattice structure.

Proposition 4.10. *Assuming the hypothesis of Proposition* 4.9, *there is a basis for*

$$\Gamma(\mathbb{P}^{k-1}, \mathscr{P}) \cong R/(J_{\mu_1} \cap \cdots \cap J_{\mu_k})$$

such that each $J_{\mu_i}/(J_{\mu_1} \cap \cdots \cap J_{\mu_k})$ is spanned by a subset of this basis.

We additionally make the following observations. We notice that certain pieces of the direct sum decomposition of $\Gamma(\mathbb{P}^{k-1}, \mathscr{P})$ correspond to certain exterior powers of $\mathscr{Q}$, in other words to certain intersection of the spaces $\boldsymbol{H}_{\mu_1} + \cdots + \boldsymbol{H}_{\mu_k}$. For example, the basis elements which belong to the trivial piece, $\Gamma(\mathbb{P}^{k-1}, \Lambda^0 \mathscr{Q}^{n!/k}) = (\Lambda^0 V)^{n!/k}$, correspond to $R/(J_{\mu_1} + \cdots + J_{\mu_k}) \cong \boldsymbol{H}_{\mu_1} \cap \cdots \cap \boldsymbol{H}_{\mu_k}$ since the basis elements coming from the trivial piece remain linearly independent in every fiber, $\mathscr{P}_{I_{\mu_1}}, \ldots, \mathscr{P}_{I_{\mu_k}}$. A basis element coming from $\Gamma(\mathbb{P}^{k-1}, \Lambda^1 \mathscr{Q}) = V$ has the property that it is mapped to zero in exactly one of the fibers $\mathscr{P}_{I_{\mu_1}}, \ldots, \mathscr{P}_{I_{\mu_k}}$, e.g., $b_i \in V$ is mapped to zero in the fiber $\mathscr{P}_{I_{\mu_i}}$ and no other fiber $\mathscr{P}_{I_{\mu_j}}$ for $j \neq 1$. Thus, these basis elements in the intersection of $k-1$ modules. In general, a basis element coming from $\Gamma(\mathbb{P}^{k-1}, \Lambda^j \mathscr{Q}) = \Lambda^j V$ corresponds to the intersection of $k - j$ of these spaces. For example, the basis element of the form $b_{i_1} \wedge \cdots \wedge b_{i_j} \in \Lambda^j V$ maps to zero in exactly the fibers with the same indices, $\mathscr{P}_{I_{\mu_{i_1}}}, \ldots, \mathscr{P}_{I_{\mu_{i_j}}}$.

In order to explain the geometric versions of Conjecture 2.5 first we observe that there is an equivariant S_n action on $\mathscr{P}$ induced from the action on $\mathscr{B}^{\otimes n}$, and an equivariant T^2 action coming from the action on $\mathrm{Hilb}^n(\mathbb{A}^2)$. These actions commute. Thus,

$\mathscr{P}$ is vector bundle of S_n modules and the action of T^2 gives it a double grading. We can write $\mathscr{P}|_{\mathbb{P}^{k-1}}$ as

$$\mathscr{P}|_{\mathbb{P}^{k-1}} \cong (\Lambda^0 \mathscr{Q} \otimes V_{(0)}) \oplus (\Lambda^1 \mathscr{Q} \otimes V_{(1)}) \oplus \cdots \oplus (\Lambda^{k-1} \mathscr{Q} \otimes V_{(k-1)}), \tag{4.13}$$

where in any particular fiber, $V_{(i)}$ is an $n!/k\binom{k-1}{i}$ dimensional doubly graded S_n module. On the level of polynomials in R representing elements of $\Gamma(\mathbb{P}^{k-1}, \mathscr{P})$, the T^2 action gives the bidegrees and the S_n action permutes the indices.

Now, recall the pairing $\mathscr{B}^{\otimes n} \otimes \mathscr{B}^{\otimes n} \to \mathscr{O}(1)$ given in (3.2) induces a perfect pairing $\mathscr{P} \otimes \mathscr{P} \to \mathscr{O}(1)$, which is equivariant under the two group actions. This pairing also gives an equivariant isomorphism $\mathscr{P} \cong \mathscr{P}^* \otimes \mathscr{O}(1)$. Furthermore, it can be shown that the perfect pairing, $\mathscr{P} \otimes \mathscr{P} \to \mathscr{O}(1)$, must induce a perfect pairing

$$\Lambda^j \mathscr{Q} \otimes \Lambda^{k-1-j} \mathscr{Q} \to \mathscr{O}(1)|_{\mathbb{P}^{k-1}},$$

where $\Lambda^j \mathscr{Q}$ and $\Lambda^{k-1-j} \mathscr{Q}$ appear as summands in the decomposition of $\mathscr{P}|_{\mathbb{P}^{k-1}}$.

We observe that the T^2 character of $(\Lambda^n \mathscr{B})_{I_{\mu_i}} \cong \mathscr{O}(1)_{I_{\mu_i}}$ is $t^{n(\mu_i)} q^{n(\mu_i')}$ and S_n character is the sign character. Thus, to explain the Ψ_i operator defined in (2.11), we note that the perfect pairing defines an isomorphism

$$(\Lambda^j \mathscr{Q})_{I_{\mu_i}} \cong ((\Lambda^{k-1-j} \mathscr{Q})^* \otimes \Lambda^n \mathscr{B})_{I_{\mu_i}},$$

where the dual corresponds to inverting the parameters q and t, and tensoring with $\Lambda^n \mathscr{B} \cong \mathscr{O}(1)$ corresponds to multiplying by $t^{n(\mu_i)} q^{n(\mu_i')}$ and applying the operator ω to the S_n character. Thus it follows on the Frobenius characteristic level that

$$\mathscr{F}((\Lambda^j \mathscr{Q})_{I_{\mu_i}}) = \Psi_i(\mathscr{F}((\Lambda^{k-1-j} \mathscr{Q})_{I_{\mu_i}})).$$

In summary, the $\Gamma(\mathbb{P}^{k-1}, \mathscr{P})$ represents $\sum \boldsymbol{H}_{\mu_i}$ and each fiber $\mathscr{P}_{I_{\mu_i}}$ corresponds to one of the modules $\boldsymbol{H}_{\mu_i}$. Partial fibers of a particular summand in $\mathscr{P}|_{\mathbb{P}^{k-1}}$ correspond to the modules $\boldsymbol{H}_S$. The pairing implies that for $i \in S$,

$$\mathscr{F}(\boldsymbol{H}_S) = \Psi_i(\boldsymbol{H}_{S^c \cup \{i\}}).$$

This relation is with respect to the $T^2 \times S_n$ representation of $\Lambda^n \mathscr{B} \cong \mathscr{O}(1)$ whose character is given by $t^{n(\mu_i)} q^{n(\mu_i')}$ for T^2 and the sign character for S_n.

To explain Conjecture 2.6, we observe that this is a simple consequence of the isomorphism in Proposition 4.9.

Corollary 4.11. *Let $\mu \vdash (n+1)$ with k corners. Let $\mu_1, \mu_2, \ldots, \mu_k \vdash n$, where μ_i is μ with the ith corner removed. Then* (*under the hypothesis of Proposition* 4.9) *for* $\{i_1, \ldots, i_m\} \subseteq \{1, \ldots, k\}$,

$$\dim(\boldsymbol{H}_{\mu_{i_1}} \cap \cdots \cap \boldsymbol{H}_{\mu_{i_m}}) = \frac{n!}{m}.$$

Finally, we mention that the Decomposition Conjecture, which gave the multiplicities of each exterior power of $\mathscr{Q}$, geometrically explained the conjuectures of Bergeron and Garsia. However, we note that the pairing relation, or equivalently, $\mathscr{P} \cong \mathscr{P}^* \otimes \mathscr{O}(1)$ always holds if $\mathscr{P}$ is assumed to be a vector bundle. Assuming the decomposition of $\mathscr{P}$ into exterior powers of $\mathscr{Q}$ without knowing the multiplicities, we can use the pairing relation along with the theory of Macdonald polynomials to still arrive at the multiplicities. We plan to discuss this in a separate paper.

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